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Convergence in Approximation and Nonsmooth Analysis

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1. INTRODUCTION

In this paper we examine some stability questions associated with problems in approximation theory and nonsmooth analysis. We study what happens to various sets that often appear in those fields, when we perturb in some sense the data determining them. Such sensitivity analysis is important for developing efficient numerical algorithms. In the next section we investigate the continuity of the map $A \to P_A(x)$, where $P_A(x)$ is the set of all best approximations to x from A. This problem was first considered by Brosowski *et al.* [5], who considered a family $\{A_t\}_{t\in T}$ of subsets of a normed linear space X parametrized by a topological space T and studied the continuity of $t \to P_{A_t}(x)$. Recently Tsukada [26] addressed the same problem but with a nonparametrized method. Namely, he allowed the sets $\{A_n\}_{n\geqslant 1}$ to converge in some sense to A, and he examined what happens to the sequence $\{P_{A_n}(x)\}_{n\geqslant 1}$. However, he limited himself to reflexive, strictly convex, smooth Banach spaces in which case the set $P_A(x)$ for A, a nonempty, closed, convex subset of X is a singleton (i.e., A is a Chebyshev set). Our work, on the one hand, generalizes the work of Tsukada [26] and, on the other hand, provides several new results on this issue. In Section 3, we investigate analogous questions in the context of nonsmooth analysis. Finally, in Section 4, we pass to prediction sequences in the Lebesgue–Bochner space $L^1_X(\Omega, \Sigma)$. Here we examine the convergence of those sequences as we vary the sub-σ-field of Σ, not necessarily in a monotone way, and the function $f(\cdot)$ that has to be approximated.

Our basic tool in this study will be Kuratowski–Mosco convergence of sets and the corresponding τ-convergence of functions. So let $\{A_n\}_{n\geqslant 1}$ be a

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sequence of subsets of a Banach space X. Define the weak limit superior of the sequence $\{A_n\}_{n\geq 1}$ to be the set

$$w-\limsup_{n\to\infty} A_n = \left\{x\in X: x = w-\lim_{k\to\infty} x_{n_k}, x_{n_k}\in A_{n_k}, k\geq 1\right\}$$

and the strong limit inferior of $\{A_n\}_{n\geq 1}$ to be the set

$$s-\liminf_{n\to\infty} A_n = \left\{x\in X: x = s-\lim_{n\to\infty} x_n, x_n\in A_n, n\geq 1\right\}.$$

We will say that A_n converges to A in the Kuratowski–Mosco sense if and only if

$$w-\limsup_{n\to\infty} A_n \subseteq A \subseteq s-\liminf_{n\to\infty} A_n.$$

Since we always have that $s-\liminf_{n\to\infty} A_n \subseteq w-\limsup_{n\to\infty} A_n$, we deduce that A_n converges to A in the Kuratowski–Mosco sense if and only if $w-\limsup_{n\to\infty} A_n = s-\liminf_{n\to\infty} A_n = A$. Then we write $A_n \to^{K-M} A$, as $n\to\infty$.

Using this set convergence we can define a new mode of convergence for extended real valued functions. So let $\{f_n, f\}_{n\geq 1}\subseteq \bar{\mathbb{R}}^X$. We say that f_n τ-converges to f if and only if epi $f_n \to^{K-M}$ epi f, as $n\to\infty$. In general, τ-convergence is not comparable to pointwise convergence. These convergence concepts were introduced by Mosco [14] and were extensively studied by Salinetti and Wets [23, 24].

We will close this introductory section by recalling a few basic facts about measurable multifunctions that we will need in the sequel. Assume (Ω, Σ, μ) is a finite complete measure space and X a separable Banach space. We use the following notations:

$$P_f(X) = \{A\subseteq X: \text{nonempty, closed}\}$$

$$P_{(w)fc}(X) = \{A\subseteq X: \text{nonempty, (weakly) closed, convex}\}$$

$$P_{(w)kc}(X) = \{A\subseteq X: \text{nonempty, (weakly) compact, convex}\}.$$

A multifunction F: $\Omega\to P_f(X)$ is said to be (weakly) measurable if it satisfies any of the following equivalent conditions:

(1) $F^-(U) = \{\omega\in\Omega: F(\omega)\cap U\neq\emptyset\}\in\Sigma$, for all $U\subseteq X$ open;

(2) For all $x\in X$, the map $\omega\to d_{F(\omega)}(x) = \inf_{y\in F(\omega)} \|x-y\|$;

(3) There exists a sequence $\{f_n(\cdot)\}_{n\geq 1}$ of measurable selectors f_n: $\Omega\to X$ of $F(\cdot)$ s.t. $F(\omega) = cl\{f_n(\omega)\}_{n\geq 1}$, for all $\omega\in\Omega$ (Castaing's representation).

Consider the set $S_F^1 = \{f(\cdot) \in L_X^1(\Omega): f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$, i.e. S_F^1 contains all integrable selectors of $F(\cdot)$. Clearly S_F^1 is a closed subset of $L_X^1(\Omega)$ and is nonempty if and only if $\inf_{y \in F(\omega)} \|y\| \in L^1(\Omega)$. Using this set we can now define an integral for $F(\cdot)$ as follows:

$$\int_\Omega F(\omega)\, d\mu(\omega) = \left\{ \int_\Omega f(\omega)\, d\mu(\omega): f(\cdot) \in S_F^1 \right\},$$

where $\int_\Omega f(\omega)\ d\mu(\omega)$ is the usual Bochner integral. This integral was introduced by Aumann [3] as a natural generalization of the Minkowski sum of sets and of the integral of a single valued function. For a comprehensive treatment of measurable multifunctions we refer to Himmelberg [8] and Rockafellar [21, 22].

A final piece of terminology: If $f \in \overline{\mathbb{R}}^X$, we define the effective domain of $f(\cdot)$ to be the set $\operatorname{dom} f = \{x \in X: f(x) < +\infty\}$.

2. Convergence in Approximation

For the first result assume that X is a reflexive Banach space and that $\{A_n, A\}_{n \geq 1} \subseteq P_{fc}(X)$.

Proposition 2.1. *If* $A_n \to^{K-M} A$, *as* $n \to \infty$, $x \in X$, *then for all* $\{x_n\}_{n \geq 1} \subseteq X$ s.t., $x_n \to x$, we have $d_{A_n}(x_n) \to d_A(x)$.

Proof. Observe that $|d_{A_n}(x_n) - d_A(x)| \leq |d_{A_n}(x_n) - d_{A_n}(x)| + |d_{A_n}(x) - d_A(x)| \leq \|x_n - x\| + |d_{A_n}(x) - d_A(x)|$. From Theorem 2.5(i) of Tsukada [26], we know that $d_{A_n}(\cdot) \to d_A(\cdot)$. So passing to the limit as $n \to \infty$, we get that $\lim_{n \to \infty} d_{A_n}(x_n) = d_A(x)$. Q.E.D.

The next result examines the stability of the best approximations to x from a sequence of sets that converges in the Kuratowski–Mosco sense. So assume that X is a Banach space and that $\{A_n, A\}_{n \geq 1} \subseteq P_f(X)$.

Proposition 2.2. *If* $A_n \to^{K-M} A$, *as* $n \to \infty$, *then for all* $x \in X$, *we have* $w - \overline{\lim}_{n \to \infty} P_{A_n}(x) \subseteq P_A(x)$.

Proof. Assume that $w - \overline{\lim}\ P_{A_n}(x) \neq \emptyset$ or otherwise the result is obvious. Let $h \in w - \overline{\lim}_{n \to \infty} P_{A_n}(x)$. Then there exist $h_{n_k} \in P_{A_{n_k}}(x)$ $k \geq 1$ s.t., $h_{n_k} \to^w h$, as $k \to \infty$. This means that $h \in w - \overline{\lim}_{n \to \infty} A_n = A$. We have $d_{A_{n_k}}(x) = \|x - h_{n_k}\|$, and because of the weak lower semicontinuity of the norm we can write that

$$\|x - h\| \leq \underline{\lim}_{k \to \infty} \|x - h_{n_k}\| = \underline{\lim}_{k \to \infty} d_{A_{n_k}}(x). \qquad (*)$$

On the other hand, for $z \in s-\underline{\lim}_{n \to \infty} A_n = A$ there exist $z_n \in A_n$ s.t., $z_n \to^s z$, as $n \to \infty$. Since $d_{A_n}(x) \leqslant \|x - z_n\|$, we get that $\overline{\lim}_{n \to \infty} d_{A_n}(x) \leqslant \|x - z\|$. But z was abitrary in A. Hence $\overline{\lim}_{n \to \infty} d_{A_n}(x) \leqslant d_A(x)$, combining that with $(*)$, we get that $\|x - h\| \leqslant d_A(x)$. But recall that $h \in A$. So $d_A(x) = \|x - h\|$, i.e., $h \in P_A(x)$. Therefore $w-\overline{\lim}_{n \to \infty} P_{A_n}(x) \subseteq P_A(x)$.
Q.E.D.

Recall that if X is reflexive and strictly convex then every $A \in P_{fc}(X)$ is a Chebyshev set, i.e., $P_A(x)$ is a singlěton for all $x \in X$. Also X is said to have property (**H**) if and only if for every $\{x_n\}_{n \geqslant 1} \subseteq X$ s.t., $x_n \to^w x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to^s x$. Locally uniformly convex spaces (in particular Hilbert spaces) have property (**H**). Using Proposition 2.2, we can have the following corollary which is Theorem 3.2(i) of Tsukada [26].

Corollary [26]. *If X is reflexive and strictly convex and $\{A_n, A\}_{n \geqslant 1} \subseteq P_{fc}(X)$ with $A_n \to^{K-M} A$, as $n \to \infty$, then for all $x \in X$, $P_{A_n}(x) \to^w P_A(x)$, as $n \to \infty$. If in addition X has property* (**H**) *then the convergence is strong.*

Proof. For all $n \geqslant 1$, we have $d_{A_n}(x) = \|x - P_{A_n}(x)\|$ and $d_A(x) = \|x - P_A(x)\|$. Also we know that $\|x - P_{A_n}(x)\| \to \|x - P_A(x)\|$. Hence there exists $M > 0$ s.t. $\|x - P_{A_n}(x)\| \leqslant M \Rightarrow \|P_{A_n}(x)\| \leqslant M + \|x\|$. Since X is reflexive we can find a subsequence $\{P_{A_{n_k}}(x)\}_{k \geqslant 1}$ s.t. $P_{A_{n_k}}(x) \to^w a$. From Proposition 2.2, we deduce that $a = P_A(x)$. So every subsequence of $\{P_{A_n}(x)\}_{n \geqslant 1}$ has a further subsequence that converges to $P_A(x)$. Thus $P_{A_n}(x) \to^w P_A(x)$. Now if X has property (**H**) then we have that $x - P_{A_n}(x) \to^s x - P_A(x) \Rightarrow P_{A_n}(x) \to^s P_A(x)$, as $n \to \infty$. Q.E.D.

Several proofs in the theory of best approximation involve elements $z \in A$ s.t., $\|x - z\| \leqslant d_A(x) + \varepsilon$, where $\varepsilon > 0$. We call such elements elements of ε-approximation (or following Buck, elements of good approximation). In addition, such elements are useful in designing algorithms that determine the vectors that realize the best approximation. We will denote the set of all such ε-approximations by $P^{\varepsilon}_A(x)$.

For these sets we can have a stronger stability result. Assume that X is reflexive and that $\{A_n, A\}_{n \geqslant 1} \subseteq P_{fc}(X)$.

Proposition 2.3. *If $A_n \to^{K-M} A$, as $n \to \infty$, then for all $\varepsilon > 0$ and all $x \in X$, we have that $P^{\varepsilon}_{A_n}(x) \to^{K-M} P^{\varepsilon}_A(x)$, as $n \to \infty$.*

Proof. By definition for all $n \geqslant 1$, we have that

$$\begin{aligned} P^{\varepsilon}_{A_n}(x) &= \{h \in A_n : \|x - h\| \leqslant d_{A_n}(x) + \varepsilon\} \\ &= \{h \in A_n : \|x - h\| - d_{A_n}(x) \leqslant \varepsilon\}. \end{aligned}$$

Let $\phi_n^x(h) = \|x - h\| - d_{A_n}(x) + \delta_{A_n}(h)$. Then we see that

$$P_{A_n}^{\varepsilon}(x) = \{h \in X: \phi_n^x(h) \leqslant \varepsilon\} = L_{\phi_n^x}^{\varepsilon}.$$

From Tsukada [26], we know that $d_{A_n}(\cdot) \to d_A(\cdot)$, as $n \to \infty$. Also since by hypothesis $A_n \to^{K-M} A$, we have that $\delta_{A_n} \to^{\tau} \delta_A$, as $n \to \infty$. From these two facts it is easy to see that $\phi_n^x(\cdot) \to^{\tau} \phi^x(\cdot)$. Clearly $\{\phi_n^x(\cdot), \phi^x(\cdot)\}$ are closed, convex functions. Hence Lemma 3.1 of Mosco [14] tells us that $L_{\phi_n^x}^{\varepsilon} \to^{K-M} L_{\phi^x}^{\varepsilon}$ as $n \to \infty \Rightarrow P_{A_n}^{\varepsilon}(x) \to^{K-M} P_A^{\varepsilon}(x)$ as $n \to \infty$. Q.E.D.

Additional resuls in the direction of stability in approximation theory were recently obtained by the authors in [9].

3. Convergence in Nonsmooth Analysis

In this section we investigate analogous stability problems in the more general context of nonsmooth analysis.

Recall that if X is a Banach space, by $\Gamma_0(X)$, we denote the set of all proper, convex, l.s.c., $\bar{\mathbb{R}}$-valued functions defined on X. Also if $f: X \to \bar{\mathbb{R}}$ is convex, its subdifferential at $x \in X$ is the set $\partial f(x) = \{x^* \in X^*: (x^*, z - x) \leqslant f(z) - f(x)$, for all $z \in X\}$.

The first result of this section can be viewed as an extension of Theorem 1.2 of Attouch [2]. So assume that X is reflexive, $\{f_n, f\}_{n \geqslant 1} \subseteq \Gamma_0(X)$ and that $f_n \to^{\tau} f$, as $n \to \infty$.

Proposition 3.1. *If for all $n \geqslant 1$, $x_n^* \in \partial f_n(x_n)$, $x_n^* \to^{w} x^*$ and $x_n \to^{s} x$, then $x^* \in \partial f(x)$.*

Proof. From convex analysis we know that for all $n \geqslant 1$, $x_n^* \in \partial f_n(x_n)$ if and only if $f_n(x_n) + f_n^*(x_n^*) = (x_n^*, x_n)$. Note that $(x_n^*, x_n) \to (x^*, x)$, as $n \to \infty$. Also because by hypothesis $f_n \to^{\tau} f$, from Lemma 1.1 of Mosco [14], we have that $f(x) \leqslant \underline{\lim}_{n \to \infty} f_n(x_n)$. Furthermore, since $f_n \to^{\tau} f$ implies that $f_n^* \to^{\tau^*} f^*$, as $n \to \infty$, we get that $f^*(x^*) \leqslant \underline{\lim}_{n \to \infty} f_n^*(x_n^*)$; combining those two facts with our initial observations we finally have that $f(x) + f^*(x^*) \leqslant (x^*, x)$. The Young–Fenchel inequality forces equality to hold, and so we conclude that $x^* \in \partial f(x)$. Q.E.D.

Remark. From the above result it is easy to see that for fixed $x \in X$, we have $w - \overline{\lim}\ \partial f_n(x) \subseteq \partial f(x)$. With additional hypotheses we can have a stronger result, namely, that $\partial f_n(x) \to^{K-M} \partial f(x)$. For details we refer to [18].

However, for ε-subdifferentials $\partial_{\varepsilon} f(x) = \{x^* \in X^*: (x^*, z - x) - \varepsilon \leqslant f(z) - f(x)$, for all $z \in X\}$ $\varepsilon > 0$, we can have more as the next result shows.

Assume that X is finite dimensional and that $\{f_n, f\}_{n \geq 1} \subseteq \Gamma_0(X)$, with int dom $f \neq \emptyset$.

PROPOSITION 3.2. *If $f_n \to f$, as $n \to \infty$, then for all $x \in$ dom $\partial f(\cdot) = \{z \in X\colon \partial f(z) \neq \emptyset\}$ and all $\varepsilon > 0$, we have that $\partial_\varepsilon f_n(x) \to^{K-M} \partial_\varepsilon f(x)$, as $n \to \infty$.*

Proof. For $n \geq 1$ let $g_n(h) = f_n(x+h) - f_n(x)$. Then $g_n^*(h^*) = f_n^*(h^*) + f_n(x) - (h^*, x)$. Similarly, for $g(h) = f(x+h) - f(x)$, we have that $g^*(h^*) = f^*(h^*) + f(x) - (h^*, x)$. Note that $g^*(\cdot) \geq 0$ and is equal to zero at all $h^* \in X^*$ s.t. $h^* \in \partial f(x)$. From the definition of the ε-subdifferential we get that

$$\partial_\varepsilon f_n(x) = \{h^* \in X^*\colon g_n^*(h^*) \leq \varepsilon\} = L_{g_n^*}^{\varepsilon}$$

and

$$\partial_\varepsilon f(x) = \{h^* \in X^*\colon g^*(h^*) \leq \varepsilon\} = L_{g^*}^{\varepsilon}.$$

From Corollary 2C of Salinetti and Wets [23], we know that $f_n \to^\tau f$, and this is equivalent to $f_n^* \to^{\tau^*} f^*$, as $n \to \infty$. From this we can easily see that $g_n^* \to^{\tau^*} g^*$, as $n \to \infty$. Thus Lemma 3.1 of Mosco [14] tells us that $L_{g_n^*}^{\varepsilon} \to^{K-M} L_{g^*}^{\varepsilon} \Rightarrow \partial_\varepsilon f_n(x) \to^{K-M} \partial_\varepsilon f(x)$, as $n \to \infty$. Q.E.D.

Remark. If $f_n \to^\tau f$, as $n \to \infty$, we can assume that X is a general reflexive Banach space.

In the rest of this section we will examine the continuity of some major operations of nonsmooth and multivalued analysis, with respect to the τ-convergence and the $K-M$-convergence.

We start with a result concerning asymptotic (recession) cones. Our result improves Lemma 4 of McLinden and Bergstrom [12]. So suppose that X is a Banach space and $\{A_n, A\}_{n \geq 1}$ are nonempty, convex subsets of X.

PROPOSITION 3.3. *If $A_n \to^{K-M} A$, as $n \to \infty$, then $w-\overline{\lim}_{n \to \infty} (A_n)_\infty \subseteq A_\infty$.*

Proof. Let $h \in w-\overline{\lim}\,(A_n)_\infty$. Then there exist $h_{n_k} \in (A_{n_k})_\infty$, $k \geq 1$ s.t., $h_{n_k} \to^w h$. By definition $h_{n_k} + A_{n_k} \subseteq A_{n_k} \Rightarrow w-\underline{\lim}_{k \to \infty} (h_{n_k} + A_{n_k}) \subseteq w-\underline{\lim}_{k \to \infty} A_{n_k}$. But $w-\underline{\lim}_{k \to \infty} (h_{n_k} + A_{n_k}) = h + A$ and $s-\underline{\lim}_{k \to \infty} A_{n_k} \subseteq w-\underline{\lim}_{k \to \infty} A_{n_k} \subseteq w-\overline{\lim}_{k \to \infty} A_{n_k} \Rightarrow w-\underline{\lim}_{k \to \infty} A_{n_k} = A$. Thus we finally have that $h + A \subseteq A$, which means that $h \in A_\infty$. Q.E.D.

Remark. It is possible to have strict inclusion as the next example illustrates: Let $A_n = [0, n]$, $A = [0, \infty)$. Clearly, $A_n \to^{K-M} A$. But $(A_n)_\infty = \{0\}$ while $A_\infty = A = [0, \infty)$. So $\overline{\lim}_{n \to \infty} (A_n)_\infty = \{0\} \subsetneq A = [0, \infty)$.

Using the previous proposition we can now prove a corresponding result for the recession functions. Again X is a Banach space and $\{f_n, f\}_{n\geq 1}$ are proper convex functions from X into $\bar{\mathbb{R}}$.

PROPOSITION 3.4. *If* $f_n \to^{\tau} f$ *and* $x_n \to^{w} x$, *as* $n \to \infty$, *then* $f_{\infty}(x) \leq \underline{\lim}_{n\to\infty} (f_n)_{\infty}(x_n)$.

Proof. We know that $f_n \to^{\tau} f$, if and only if, $\operatorname{epi} f_n \to^{K-M} \operatorname{epi} f$, as $n \to \infty$. Since $\{\operatorname{epi} f_n, \operatorname{epi} f\}_{n\geq 1}$ are nonempty, convex sets in $X \times \mathbb{R}$, we can apply Proposition 3.3 and get that $w-\overline{\lim}_{n\to\infty}(\operatorname{epi} f_n)_{\infty} \subseteq (\operatorname{epi} f)_{\infty}$. But recall that $(\operatorname{epi} f_n)_{\infty} = \operatorname{epi}(f_n)_{\infty}$ and $(\operatorname{epi} f)_{\infty} = \operatorname{epi} f_{\infty}$. So we have that $w-\overline{\lim}_{n\to\infty} \operatorname{epi}(f_n)_{\infty} \subseteq \operatorname{epi} f_{\infty}$, and thus is equivalent to saying that for all $x_n \to^{w} x, f_{\infty}(x) \leq \underline{\lim}_{n\to\infty} (f_n)_{\infty}(x_n)$. Q.E.D.

In [3], Aumann proved a dominated convergence theorem for his set valued integral under the assumption that $X = \mathbb{R}^n$. Here we present an infinite dimensional version of it. Recall that a multifunction $F: \Omega \to P_f(X)$ is said to be integrably bounded if and only if $|F(\omega)| = \sup_{x\in F(\omega)} \|x\|$ is an $L^1_+(\Omega)$ function.

Assume that X is reflexive and that $F_n(\cdot)$, $F(\cdot)$: $\Omega \to P_{fc}(X)$ are measurable multifunctions.

THEOREM 3.1. *If* $|F_n(\omega)| \leq \phi(\omega)$ *μ-a.e., for* $\phi(\cdot) \in L^1(\Omega)$ *and* $F_n(\omega) \to^{K-M} F(\omega)$ *μ-a.e., then* $\int_\Omega F_n(\omega)\, d\mu(\omega) \to^{K-M} \int_\Omega F(\omega)\, d\mu(\omega)$, *as* $n \to \infty$, *and* $S^1_{F_n} \to^{K-M} S^1_F$, *as* $n \to \infty$.

Proof. For $x^* \in X$, we have

$$\sigma_{\int_\Omega F_n}(x^*) = \sup_{y\in\int_\Omega F_n} (x^*, y) = \sup_{f\in S^1_{F_n}} \left(x^*, \int_\Omega f(\omega)\, d\mu(\omega)\right)$$

$$= \sup_{f\in S^1_{F_n}} \int_\Omega (x^*, f(\omega))\, d\mu(\omega).$$

Using Theorem 2.2 of Hiai and Umegaki [7], we have that

$$\sup_{f\in S^1_{F_n}} \int_\Omega (x^*, f(\omega))\, d\mu(\omega) = \int_\Omega \sup_{z\in F_n(\omega)} (x^*, z)\, d\mu(\omega) = \int_\Omega \sigma_{F_n(\omega)}(x^*)\, d\mu(\omega).$$

Hence we have that $\sigma_{\int_\Omega F_n}(x^*) = \int_\Omega \sigma_{F_n(\omega)}(x^*)$ and $\sigma_{\int_\Omega F}(x^*) = \int_\Omega \sigma_{F(\omega)}(x^*)$. Using Fatou's lemma we have that $\overline{\lim}\, \sigma_{\int_\Omega F_n}(x^*) = \overline{\lim} \int_\Omega \sigma_{F_n(\omega)}(x^*) \leq \int_\Omega \overline{\lim}\, \sigma_{F_n(\omega)}(x^*)$. Because of the reflexivity of X and the uniform boundedness by $\phi(\cdot)$, it is easy to see that $\overline{\lim}\, \sigma_{F_n(\omega)}(x^*) \leq \sigma_{w-\overline{\lim} F_n(\omega)}(x^*)$ μ-a.e. Hence $\overline{\lim}\, \sigma_{\int_\Omega F_n}(x^*) \leq \int_\Omega \sigma_{F(\omega)}(x^*) = \sigma_{\int_\Omega F}(x^*)$, for all $x^* \in X^* \Rightarrow w-\overline{\lim} \int_\Omega F_n \subseteq \int_\Omega F$. On the other hand, let $x \in \int_\Omega F$. Then

$x=\int_\Omega f(\omega)$ with $f(\cdot)\in S_F^1$. An easy application of Aumann's selection theorem provides $f_n(\cdot)\in S_{F_n}^1$ s.t. $d_{F_n(\omega)}(f(\omega))=\|f_n(\omega)-f(\omega)\|\to 0$ μ-a.e., as $n\to\infty \Rightarrow \int_\Omega f_n \to \int_\Omega f \Rightarrow x\in s-\underline{\lim}\int_\Omega F_n$. Therefore $\int_\Omega F\subseteq s-\underline{\lim}\int_\Omega F_n$. Thus we conclude that $\int_\Omega F_n \to^{K-M}\int_\Omega F$.

For the second part note that for every $u(\cdot)\in L_{X^*}^\infty=(L_X^1)^*$, we can show as above that $\sigma_{S_{F_n}^1}(u) = \int_\Omega \sigma_{F_n(\omega)}(u(\omega))$. Hence $\overline{\lim}\,\sigma_{S_{F_n}^1}(u) \leqslant \int_\Omega \overline{\lim}\,\sigma_{F_n(\omega)}(u(\omega)) \leqslant \int_\Omega \sigma_{F(\omega)}(u(\omega)) = \sigma_{S_F^1}(u) \Rightarrow w-\overline{\lim}\,S_{F_n}^1\subseteq S_F^1$. Also note that using Theorem 2.2 of [7], we can show that $d_{S_{F_n}^1}(g) = \int_\Omega d_{F_n(\omega)}(g(\omega))$, for all $g(\cdot)\in L_X^1$. Therefore using Theorem 2.2 (i) and (ii) of Tsukada [26], we get that $\overline{\lim}\,d_{S_{F_n}^1}(g) \leqslant \int_\Omega \overline{\lim}\,d_{F_n(\omega)}(g(\omega)) \leqslant \int_\Omega d_{F(\omega)}(g(\omega)) = d_{S_F^1}(g) \Rightarrow S_F^1\subseteq s-\underline{\lim}\,S_{F_n}^1$. So finally, we have that $S_{F_n}^1 \to^{K-M} S_F^1$. Q.E.D.

Before stating the last result of this section we need to present an easy lemma. For normal integrals see Rockafellar [22]. Here X is any separable Banach space.

LEMMA. *If $f\colon \Omega\times X\to\mathbb{R}$ is a convex, normal integrand s.t., for all $x\in X$, $f(\cdot,x)$ and $[f^*(\cdot,0)]^+$ have finite integrals, then for $\Phi(x)=\int_\Omega f(\omega,x)\,d\mu(\omega)$, we have that $\Phi_\infty(h)=\int_\Omega f_\infty(\omega,h)\,d\mu(\omega)$, $h\in X$.*

Proof. From Proposition 6.8.3 of Laurent [1], we have that for any x, $h\in X$,

$$f_\infty(\omega,h)=\sup_{\lambda>0}\frac{f(\omega,x+\lambda h)-f(\omega,x)}{\lambda}$$

$$\Rightarrow \int_\Omega f_\infty(\omega,h)\,d\mu(\omega)=\int_\Omega \sup_{\lambda>0}\frac{f(\omega,x+\lambda h)-f(\omega,x)}{\lambda}\,d\mu(\omega),$$

but the integrands of the right hand side increase with $\lambda>0$. So an application of the monotone convergence theorem gives us

$$\int_\Omega f_\infty(\omega,h)\,d\mu(\omega)=\sup_{\lambda>0}\int_\Omega \frac{f(\omega,x+\lambda h)-f(\omega,x)}{\lambda}\,d\mu(\omega)$$

$$=\sup_{\lambda>0}\frac{\Phi(x+\lambda h)-\Phi(x)}{\lambda}.$$

Note that $\Phi(\cdot)$ is finite convex. Also recall that $-f^*(\omega,0)=\inf_{x\in X} f(\omega,x)$, and since by hypothesis $[f^*(\cdot,0)]^+$ is integrable, we deduce that for all $x\in X$, $f(\cdot,x)$ is bounded from below by a function whose integral is bigger than $-\infty$. Hence if $x_n \to x$, an application of Fatou's lemma gives us that

$$\Phi(x) = \int_\Omega f(\omega, x)\, d\mu(\omega) \leqslant \int_\Omega \varliminf_{n\to\infty} f(\omega, x_n)\, d\mu(\omega)$$

$$\leqslant \varliminf_{n\to\infty} \int_\Omega f(\omega, x_n)\, d\mu(\omega) = \varliminf_{n\to\infty} \Phi(x_n).$$

Therefore $\Phi(\cdot)$ is in $\Gamma_0(X)$ and so $\sup_{\lambda>0} (\Phi(x+\lambda h) - \Phi(x))/\lambda = \Phi_\infty(h)$. Thus we have shown that $\int_\Omega f_\infty(\omega, h)\, d\mu(\omega) = \Phi_\infty(h)$, for all $h \in X$. Q.E.D.

We will use that result to study the convergence of the recession function of the integral functional $\Phi(\cdot)$. So assume that X is a finite dimensional Banach space.

PROPOSITION 3.5. *If $f_n, f\colon \Omega \times X \to \mathbb{R}$ are normal, convex integrands s.t., for all $x \in X$, $f_n(\cdot, x) \to^{w-L^1} f(\cdot, x)$ and $[f_n^*(\cdot, 0)]^+$, $[f^*(\cdot, 0)]^+$ are integrable, then for all $x_n \to^{w} x$, we have that*

$$\int_\Omega f_\infty(\omega, x)\, d\mu(\omega) \leqslant \varliminf_{n\to\infty} \int_\Omega (f_n)_\infty(\omega, x_n)\, d\mu(\omega).$$

Proof. Since by hypothesis for all $x \in X$, $f_n(\cdot, x) \to^{w-L^1} f(\cdot, x)$, we have that

$$\int_\Omega f_n(\omega, x)\, d\mu(\omega) \to \int_\Omega f(\omega, x)\, d\mu(\omega),$$

i.e.,

$$\Phi_n(x) \to \Phi(x), \text{ as } n \to \infty, \text{ for all } x \in X.$$

Observe that $\{\Phi_n(\cdot), \Phi(\cdot)\}_{n\geqslant 1} \subseteq \Gamma_0(X)$ and $\operatorname{dom} \Phi_n = \operatorname{dom} \Phi = X$. Thus Corollary 2C of Salinetti and Wets [23] tells us that $\Phi_n(\cdot) \to^{\tau} \Phi(\cdot)$. Hence we can apply Proposition 3.4 and deduce that $\Phi_\infty(x) \leqslant \varliminf_{n\to\infty} (\Phi_n)_\infty(x_n)$. Using the lemma, we finally can say that $\int_\Omega f_\infty(\omega, x)\, d\mu(\omega) \leqslant \varliminf_{n\to\infty} \int_\Omega (f_n)_\infty(\omega, x_n)\, d\mu(\omega)$. Q.E.D.

We will conclude this section with a result that establishes the continuity of the intersection operation with respect to the $K-M$-convergence. So assume that $X = \mathbb{R}^n$ and that $\{A_n, B_n, A, B\} \subseteq P_{fc}(X)$.

PROPOSITION 3.6. *If for all $n \geqslant 1$, $\operatorname{int} A_n \neq \varnothing$, $\operatorname{int} A \neq \varnothing$, $A_n \to^{K-M} A$, $B_n \to^{K-M} B$, as $n \to \infty$, and $\operatorname{int} A_n \cap B \neq \varnothing$, $\operatorname{int} A \cap B \neq \varnothing$, then $A_n \cap B_n \to^{K-M} A \cap B$, as $n \to \infty$.*

Proof. From Lemma 5 of Moreau [13], we know that for all $n \geq 1$,

$$\sigma_{A_n \cap B_n}(\cdot) = (\sigma_{A_n} \,\square\, \sigma_{B_n})(\cdot)$$

and

$$\sigma_{A \cap B}(\cdot) = (\sigma_A \,\square\, \sigma_B)(\cdot),$$

where $\square$ denotes the operation of infimal convolution (see Laurent [11] or Rockafellar [20]). Since by hypothesis $\operatorname{int} A \cap B \neq \varnothing$, we have that $0 \in \operatorname{int}(A - B)$, and this is equivalent to saying that $(\operatorname{epi} \sigma_A)_\infty \cap (-\operatorname{epi} \sigma_B)_\infty = \{0\}$. Hence we can apply Theorem 4 of McLinden and Bergstrom [12] and get that

$$\begin{aligned} &\sigma_{A_n} \,\square\, \sigma_{B_n} \xrightarrow{\tau^*} \sigma_A \,\square\, \sigma_B \\ &\Rightarrow \sigma_{A_n \cap B_n} \xrightarrow{\tau^*} \sigma_{A \cap B} \\ &\Rightarrow A_n \cap B_n \xrightarrow{K-M} A \cap B, \text{ as } n \to \infty. \end{aligned}$$

Q.E.D.

4. Prediction Sequences in L^1_X

Suppose that (Ω, Σ, μ) is a complete probability space and that X is a separable reflexive Banach space. Let Σ_0 be a sub-σ-field of Σ. Consider the Lebesgue–Bochner space $L^1_X(\Sigma_0)$ and let $f(\cdot) \in L^1_X(\Sigma) = L^1_X$. We define

$$d_{\Sigma_0}(f) = \inf_{g \in L^1_X(\Sigma_0)} \|f - g\|_1$$

and

$$P_{\Sigma_0}(f) = \{g \in L^1_X(\Sigma_0) \colon d_{\Sigma_0}(f) = \|f - g\|_1\}.$$

The main purpose of this section is to study the behavior of those two operators under variations of the sub-σ-field Σ_0 and of the function $f(\cdot)$.

We start with an existence result analogous to Theorem 5 of Shintani and Ando [25], which was for the case $X = \mathbb{R}$. However, thanks to the reflexivity of X and the Dunford–Pettis compactness criterion (see Diestel and Uhl [6], p. 101), their proof also applies to this more general case. So we have:

Proposition 4.1. *Every minimizing sequence is relatively w-compact and its w-limits are in $P_{\Sigma_0}(f)$. Hence in particular, $L^1_X(\Sigma_0)$ is proximinal.*

In the sequel we will need the following lemma.

LEMMA 4.1. *If $g(\cdot)\in P_{\Sigma_0}(f)$, then $\|g(\omega)\|\leqslant 2E^{\Sigma_0}\|f(\omega)\|$ μ a.e.*

Proof. Let $A\in\Sigma_0$. Observing that $\chi_{A^c}g\in L^1_X(\Sigma_0)$, we have that

$$\int_A \|g(\omega)\|\,d\mu(\omega)+d_{\Sigma_0}(f)$$

$$\leqslant\int_A \|g(\omega)\|\,d\mu(\omega)+\int_\Omega \|f(\omega)-\chi_{A^c}(\omega)\,g(\omega)\|\,d\mu(\omega)$$

$$\leqslant\int_A \|f(\omega)-g(\omega)\|\,d\mu(\omega)+\int_A \|f(\omega)\|\,d\mu(\omega)$$

$$+\int_\Omega \|f(\omega)-\chi_{A^c}(\omega)\,g(\omega)\|\,d\mu(\omega)$$

$$=\int_A \|f(\omega)-g(\omega)\|\,d\mu(\omega)+\int_A \|f(\omega)\|\,d\mu(\omega)$$

$$+\int_A \|f(\omega)\|\,d\mu(\omega)+\int_{A^c} \|f(\omega)-g(\omega)\|\,d\mu(\omega)$$

$$=\int_\Omega \|f(\omega)-g(\omega)\|\,d\mu(\omega)+2\int_A \|f(\omega)\|\,d\mu(\omega)$$

$$=d_{\Sigma_0}(f)+2\int_A \|f(\omega)\|\,d\mu(\omega).$$

Hence we have that

$$\int_A \|g(\omega)\|\,d\mu(\omega)\leqslant 2\int_A \|f(\omega)\|\,d\mu(\omega)=2\int_A E^{\Sigma_0}\|f(\omega)\|\,d\mu(\omega).$$

Since this is true for all $A\in\Sigma_0$, we conclude that $\|g(\omega)\|\leqslant 2E^{\Sigma_0}\|f(\omega)\|$ μ-a.e. Q.E.D.

Now we will introduce the convergence of σ-fields that we are going to use through our sensitivity analysis. So let $\{\Sigma_n\}$ be a sequence of sub-σ-fields of Σ. We will say that Σ_n converges in L^1_X to Σ_∞ also a sub-σ-field of Σ if and only if, for all $f\in L^1_X$, $E^{\Sigma_n}f\to^{L^1_X}E^{\Sigma_\infty}f$. In [17] the first author proved that if $\limsup_{n\to\infty}\Sigma_n=\liminf_{n\to\infty}\Sigma_n=\Sigma_\infty$, then $\Sigma_n\to^{L^1_X}\Sigma_\infty$. Also the martingale convergence theorem (see for example Neveu [16], Proposition V-2-6, p. 104) tells us that if $\{\Sigma_n\}_{n\geqslant 1}$ is monotone increasing, then $\Sigma_n\to^{L^1_X}\Sigma_\infty$, as $n\to\infty$. When $X=\mathbb{R}$, this convergence is in fact equivalent to the strong convergence (s-convergence) of σ-fields introduced by Neveu [15] and studied by Kudo [10] and Becker [4].

The first result tells us that the subspaces $L^1_X(\Sigma_n)$ are continuous in a certain sense with respect to the above convergence of σ-fields. For this result, X is any separable Banach space with X^* having the Radon–Nikodym property (R.N.P.).

LEMMA 4.2. *Let* $\{f_n\} \subseteq L^1_X$ *and* $\{g_n\}_{n\geq 1} \subseteq L^\infty_{X^*}$ s.t., $f_n \to^{w-L^1_X} f$, $\sup_{n\geq 1} \|g_n\|_\infty \leq M$, *and* $g_n \to^{s-L^1_{X^*}} g \in L^\infty_{X^*}$. *Then* $\langle f_n, g_n\rangle \to \langle f, g\rangle$.

Proof. Without loss of generality assume that $g=0$. Then for $a>0$, we have

$$|\langle f_n, g_n\rangle| \leq \int_\Omega |(f_n(\omega), g_n(\omega))|\, d\mu(\omega)$$

$$\leq \int_{\{\|f_n\|>a\}} f_n(\omega)\| \, \|g_n(\omega)\|\, d\mu(\omega) + \int_{\{\|f_n\|\leq a\}} \|f_n(\omega)\| \, \|g_n(\omega)\|\, d\mu(\omega)$$

$$\leq M \int_{\{\|f_n\|>a\}} \|f_n(\omega)\|\, d\mu(\omega) + a\|g_n\|_1;$$

but $\{f_n\}_{n\geq 1}$ is w-compact in L^1_X, so uniformly integrable. Hence

$$\sup_{n\geq 1} \int_{\{\|f_n\|>a\}} \|f_n(\omega)\|\, d\mu(\omega) \to 0, \text{ as } a\to\infty \Rightarrow \langle f_n, g_n\rangle \to 0. \quad \text{Q.E.D.}$$

Using this lemma we can prove the following.

PROPOSITION 4.2. *If* $\Sigma_n \to^{L^1_X} \Sigma_\infty$, *then* $L^1_X(\Sigma_n) \to^{K-M} L^1_X(\Sigma_\infty)$.

Proof. Let $f \in w-\overline{\lim}\, L^1_X(\Sigma_n)$. Then there exists $f_k(\cdot) \in L^1_X(\Sigma_{n_k})$ s.t. $f_k \to^{w-L^1_X} f$. For any $g(\cdot) \in L^\infty_{X^*}$, using that $\Sigma_n \to^{L^1_X} \Sigma_\infty$ and Lemma 4.2, we have

$$\langle f, g\rangle = \lim_{k\to\infty} \langle f_k, g\rangle = \lim_{k\to\infty} \langle f_k, E^{\Sigma_k} g\rangle = \langle f, E^{\Sigma_\infty} g\rangle.$$

So $f = E^{\Sigma_\infty} f \Rightarrow f \in L^1_X(\Sigma_\infty) \Rightarrow w-\overline{\lim}\, L^1_X(\Sigma_n) \subseteq L^1_X(\Sigma_\infty)$. On the other hand, let $f \in L^1_X(\Sigma_\infty)$. Then $E^{\Sigma_n} f \in L^1_X(\Sigma_n)$ and $E^{\Sigma_n} f \to^{s-L^1_X} E^{\Sigma_\infty} f = f$. So $L^1_X(\Sigma_\infty) \subseteq s-\underline{\lim}\, L^1_X(\Sigma_n)$. Thus, finally, $L^1_X(\Sigma_n) \to^{K-M} L^1_X(\Sigma_\infty)$. Q.E.D.

Now we are ready for the main result of this section. So assume that X is also reflexive.

THEOREM 4.1. *If* $\Sigma_n \to^{L^1_X} \Sigma_\infty$ *and* $f_n \to^{L^1_X} f$, *as* $n\to\infty$, *then any sequence* $g_n \in P_{\Sigma_n}(f_n)$ *is relatively weakly compact in* L^1_X; *its weak limits are in* $P_{\Sigma_\infty}(f)$ *and* $d_{\Sigma_n}(f_n) \to d_{\Sigma_\infty}(f)$, *as* $n\to\infty$.

Proof. From Lemma 4.1 we know that for all $n \geqslant 1$, $\|g_n\|_1 \leqslant 2\|f_n\|_1$. But $\{f_n\}_{n\geqslant 1}$ is an L^1_X-convergent sequence and so it is L^1_X-bounded and uniformly integrable (see Diestel and Uhl [6], p. 104). Furthermore, since X is reflexive for all $A \in \Sigma$, $K_A = \{\int_A g_n(\omega)\, d\mu(\omega)\}_{n\geqslant 1}$ is a relatively weakly compact subset of X. So invoking the Dunford–Pettis compactness criterion, we deduce that $\{g_n\}_{n\geqslant 1}$ is relatively w-compact in L^1_X. Thanks to the Eberlein–Smulian theorem, we know that it is relatively w-sequentially compact. So let $g_{n_k} \xrightarrow{w-L^1_X} g$. We will show that $g \in P_{\Sigma_\infty}(f)$. From the weak lower semicontinuity of the norm we have that $\|f-g\|_1 \leqslant \underline{\lim}_{k\to\infty} \|f-g_{n_k}\|_1$. Let $\hat{g} \in P_{\Sigma_\infty}(f)$. From Proposition 4.1, we know that such a function exists. Then we have

$$\begin{aligned}\|f-g\|_1 &\leqslant \underline{\lim}_{k\to\infty}\, d_{\Sigma_{n_k}}(f) \leqslant \underline{\lim}_{k\to\infty}\, \|f - E^{\Sigma_{n_k}}\hat{g}\|_1 \\ &\leqslant \|f-\hat{g}\|_1 + \underline{\lim}_{k\to\infty}\, \|\hat{g} - E^{\Sigma_{n_k}}\hat{g}\|_1 \\ &= d_{\Sigma_\infty}(f) + \underline{\lim}_{k\to\infty}\, \|E^{\Sigma_\infty}\hat{g} - E^{\Sigma_{n_k}}\hat{g}\|_1,\end{aligned}$$

since by hypothesis $\Sigma_{n_k} \xrightarrow{L^1_X} \Sigma_\infty$, we have that

$$\underline{\lim}_{k\to\infty}\, \|E^{\Sigma_\infty}\hat{g} - E^{\Sigma_{n_k}}\hat{g}\|_1 = 0.$$

Thus, finally, we get that $\|f-g\|_1 \leqslant d_{\Sigma_\infty}(f)$. Since $g(\cdot) \in L^1_X(\Sigma_\infty)$, we conclude that $\|f-g\|_1 = d_{\Sigma_\infty}(f)$, i.e., $g \in P_{\Sigma_\infty}(f)$.

Next we claim that $d_{\Sigma_n}(f) \to d_{\Sigma_\infty}(f)$, as $n \to \infty$. Let $u_n \in P_{\Sigma_n}(f)$ $n \geqslant 1$. Then $d_{\Sigma_n}(f) = \|f-u_n\|_1$. From the first part of the proof we know that $\{u_n\}_{n\geqslant 1}$ is relatively w-compact in $L^1_X(\Sigma)$. Hence $u_{n_k} \xrightarrow{w-L^1_X} u \in P_{\Sigma_\infty}(f)$, as $k \to \infty \Rightarrow \underline{\lim}_{k\to\infty} d_{\Sigma_{n_k}}(f) = \underline{\lim}_{k\to\infty} \|f-u_{n_k}\|_1 \geqslant \|f-u\|_1 = d_{\Sigma_\infty}(f)$. On the other hand, from Theorem 3.2 of Tsukada [26] and Proposition 4.2, we have that $\overline{\lim}_{k\to\infty} d_{\Sigma_{n_k}}(f) \leqslant d_{\Sigma_\infty}(f)$. So $d_{\Sigma_{n_k}}(f) \to d_{\Sigma_\infty}(f)$. Therefore we have shown that every subsequence of $\{d_{\Sigma_n}(f)\}_{n\geqslant 1}$ has a further subsequence that converges to $d_{\Sigma_\infty}(f)$, which means that $d_{\Sigma_n}(f) \to d_{\Sigma_\infty}(f)$, as $n \to \infty$. Now because the distance function is Lipschitz, we have

$$\begin{aligned}&|d_{\Sigma_n}(f_n) - d_{\Sigma_\infty}(f)| \leqslant \|f_n - f\|_1 + |d_{\Sigma_n}(f) - d_{\Sigma_\infty}(f)| \\ &\quad\Rightarrow \lim_{n\to\infty} |d_{\Sigma_n}(f_n) - d_{\Sigma_\infty}(f)| = 0, \quad \text{i.e. } d_{\Sigma_n}(f_n) \to d_{\Sigma_\infty}(f), \text{ as } n \to \infty.\end{aligned}$$

Q.E.D.

Remarks. (1) The second claim of the theorem can not be obtained from Proposition 2.1, because we proved that proposition for reflexive Banach spaces.

(2) The above theorem tells us that if $\{\Sigma_n, \Sigma_\infty\}_{n\geqslant 1}$ are as described, then $w\text{-}\overline{\lim}_{n\to\infty} P_{\Sigma_n}(f_n) \subseteq P_{\Sigma_\infty}(f)$.

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